§6.6 Anomalies and Goldstone Bosons

$SU(3) \times SU(3)$ global symmetry of QCD is spontaneously broken to $SU(3)$ isospin

→ 't Hooft anomaly matching gives following spectrum in UV:

- weakly gauge $SU(3) \times SU(3)$ symmetry
  → anomalies must be cancelled by fictitnous spectator fermions
- massless fermions and $SU(N_c)$ gauge bosons

while the <u>massless</u> spectrum in IR is:

- fictitnous gauge bosons and spectator fermions
- a set of Goldstone boson fields ξ_a
  (fermionic bound states are massive due to spontaneous symmetry breaking)

→ anomaly of effective field theory of Goldstone bosons must reproduce the original UV anomaly:

$$\delta_\beta(x)\, \Gamma[\xi, A] = G_\beta[x; A]$$

β an $\mathcal{T}_\beta$ runs over values i labelling generators Y_i of unbroken symmetry H, <u>and</u> values a labelling broken symmetry generators X_a (associated with Goldstone bosons ξ_a)

<u>Some details of Goldstone bosons:</u>

symmetry breaking $G \longrightarrow H \subset G$

$$\sum_m \underset{H}{h_{nm}} \langle \phi_m(x) \rangle_{VAC} = \langle \phi_n(x) \rangle_{VAC} \qquad (1)$$

write $\phi_n(x) = \sum_m \gamma_{nm}(x) \tilde{\Phi}_m(x)$

with $\gamma \in G$ and $\tilde{\Phi}$ is a field with <u>no</u> Goldstone modes (previously $\tilde{\Phi} = (0,0,0,\phi_4)$)

As $\langle \phi_m(x) \rangle_{VAC}$ is H-invariant

$\rightarrow$ can take $\tilde{\Phi} = h^{-1}\gamma^{-1}\phi$ as well as $\tilde{\Phi} = \gamma^{-1}\phi$

$\rightarrow$ γ is only defined up to right multiplication by an element of H (thus $\gamma_1 \overset{\text{equiv}}{\sim} \gamma_2$ iff. $\gamma_1 = \gamma_2 h$, $h \in H$)

$\rightarrow$ element of "right coset" G/H

Can write $g = \exp\left[i \sum_a \xi_a X_a\right] \exp\left[i \sum_i \theta_i Y_i\right]$

for general element $g \in G$ with

$$[Y_i, Y_j] = i \sum_k C_{ijk} Y_k \quad , \quad Y_i \in H$$

$$[Y_i, X_a] = i \sum_b C_{iab} X_b \quad , \quad X_a \in G/H$$

$$[X_a, X_b] = i \sum_i C_{abi} Y_i + i \sum_c C_{abc} X_c$$

$$\longrightarrow \gamma(x) = \exp\left[i \sum_a \underset{\substack{\uparrow \\ \text{Goldstone boson field}}}{\xi_a(x)} X_a \right]$$

Under symmetry G, Goldstono bosons trf. as

$$\phi(x) \longmapsto \phi'(x) = g\,\phi(x) = g\,\gamma\big(\xi(x)\big)\,\tilde{\Phi}(x)$$

$$\text{use } g\,\gamma\big(\xi(x)\big) = \underbrace{\gamma\big(\xi'(x)\big)}_{\in G/H}\, \underbrace{h\big(\xi(x), g\big)}_{\in H} \qquad (2)$$

$$\rightarrow \quad \phi'(x) = \gamma\big(\xi'(x)\big)\,\tilde{\Phi}'(x)$$

$$\tilde{\Phi}'(x) = h\big(\xi(x), g\big)\,\tilde{\Phi}(x)$$

For example, in the case of global $SU(3) \times SU(3)$ symmetry of QCD we have:

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \longmapsto \exp\left[i \sum_a \left(\theta_a^V \lambda_a + \theta_a^A \lambda_a \gamma_5 \right) \right] \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

and $\gamma(x) = \exp\left(-i \gamma_5 \sum_a \xi_a(x) \lambda_a \right)$

Under $SU(3)\times SU(3)$ transformations we have:

$$U(x) \longmapsto \exp\left(i\sum_a \lambda_a \theta_a^R\right) U(x) \exp\left(-i\sum_a \lambda_a \theta_a^L\right) \quad (3)$$

where $U(x) \equiv \exp\left(2i\sum_a \xi_a(x)\lambda_a\right)$

$\rightarrow U$ is in $(\bar{3},3)$ rep. of $SU(3)\times SU(3)$

$\rightarrow$ can construct $(SU(3)\times SU(3))$-inv. Lagrangians:

$$\mathcal{L}_{2\,deriv} = -\frac{1}{16}F^2 \mathrm{Tr}\left\{\partial_\mu U \partial^\mu U^\dagger\right\}$$

(higher deriv. terms analogous)

We are searching for anomalous term

$$\mathcal{T}_\beta(x)\,\Gamma[\xi,0] = G_\beta[x;A\xi]$$

with $\mathcal{T}_\beta(x) = \mathcal{T}_\beta^A(x) + \mathcal{T}_\beta^\xi(x)$

$$-i\,\mathcal{T}_\beta^A(x) = -\frac{\partial}{\partial x^\mu}\frac{\delta}{\delta A_{\beta\mu}(x)} - C_{\beta\gamma\alpha}A_{\gamma\mu}(x)\frac{\delta}{\delta A_{\alpha\mu}(x)}$$

$$\mathcal{T}_\beta^\xi(x)\exp(i\xi_\beta X_a) = \exp(i\xi_a(x)X_a)\,\theta_{\beta i}(x)Y_i - T_\beta \exp(i\xi_a(x)X_a)$$

(infinitesimal limit of eq. (2))

In the case of $G = SU(3) \times SU(3)$,

$\Gamma[\xi, 0]$ is known as the

Wess-Zumino - Witten term: $\Gamma[\xi, 0] = I_{WZW}[U]$

We shall construct $I_{WZW}[U]$!

think of spacetime as $S^4 = \overline{\mathbb{R}^4}$

(require $\xi_a(x) \xrightarrow{x^\mu \to \infty} c \in \mathbb{R}$ for any direction

$\rightarrow$ include infinity as a point)

Thus we have:

$$\xi_a : S^4 \longrightarrow G/H$$

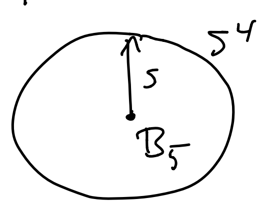
If $\pi_4(G/H) = 0$, we can continuously deform the map $\xi_a(x)$ to $\xi_a(x) = c \ \forall\, x$:

introduce $\xi_a(x; s)$ defined for $0 \geq s \geq 1$, with

$$\xi_a(x; 0) = \xi_a(x) \quad \text{and} \quad \xi_a(x; 1) = c$$

Now $\pi_4(SU(N)) = 0$ for $N \geq 3$!

$\rightarrow$ For $G = SU(3) \times SU(3)$ and $H = SU(3)$, $U(x)$ can be extended to a map $U(x)$ from $B_5 = \{(x^\mu; s) \mid x^\mu \in S^4,\ s \in [0,1]\}$ to $G/H = SU(3)$



Now consider the following function formed from $U(x)$:

$$\omega(y) \equiv -\frac{i}{240\pi^2} \varepsilon^{ijkl m} \mathrm{Tr}\left\{ U^{-1}\frac{\partial U}{\partial y^i} U^{-1}\frac{\partial U}{\partial y^j} U^{-1}\frac{\partial U}{\partial y^k} U^{-1}\frac{\partial U}{\partial y^l} U^{-1}\frac{\partial U}{\partial y^m} \right\}$$

Manifestly invariant under

$$U(x) \longmapsto \exp\left(i\sum_a \lambda_a \theta_a^R\right) U(x) \exp\left(-i\sum_a \lambda_a \theta_a^L\right)$$

Now define

$$I_{WZW}[U] = \underset{\substack{\nearrow \\ \text{so far arbitrary}}}{n} \int_{B_5} d^5y\, \omega(y)$$

I_{WZW} has following properties:

- independent of coordinate choice

$$\left(\varepsilon^{ijklm} \frac{\partial \theta'^{\bar{i}}}{\partial \theta^i} \frac{\partial \theta'^{\bar{j}}}{\partial \theta^j} \frac{\partial \theta'^{\bar{k}}}{\partial \theta^k} \frac{\partial \theta'^{\bar{l}}}{\partial \theta^l} \frac{\partial \theta'^{\bar{m}}}{\partial \theta^m} = \mathrm{Det}\left(\frac{\partial \theta'}{\partial \theta}\right) \varepsilon^{\bar{i}\bar{j}\bar{k}\bar{l}\bar{m}}\right)$$

- Only depends on values of $U(y)$ on the ball's surface S^4

$$\delta I_{WZW} = 5\int_{B_5} d^5y\, \varepsilon^{ijkl} \mathrm{Tr}\left\{ U^{-1}\frac{\partial U}{\partial y^i} \cdots U^{-1}\frac{\partial U}{\partial y^l} \delta\left(U^{-1}\frac{\partial U}{\partial y^m}\right)\right\}$$

and $\delta\left(U^{-1}\frac{\partial U}{\partial y^m}\right) = U^{-1}\frac{\partial}{\partial y^m}(\delta U\, U^{-1})\, U$

$\longrightarrow$ integrate by parts

$$\delta I_{WZW} = -\frac{i}{48\pi^2}\int_{B_5} \varepsilon^{ijklm}\frac{\partial}{\partial y^m}\mathrm{Tr}\left\{u^{-1}\frac{\partial u}{\partial y^i}\cdots u^{-1}\frac{\partial u}{\partial y^l}u^{-1}\delta u\right\}$$

$= $ surface term

Thus we can include $I_{WZW}[U]$ as a term in our 4d action.

Defining $\sum_a \lambda_a \xi_a = \frac{\sqrt{2}\,B}{F}$, we can write in the limit of small meson fields:

$$\omega(x) \longrightarrow \frac{8\sqrt{2}}{15\pi^2 F_\pi^5}\varepsilon^{ijklm}\mathrm{Tr}\left\{\frac{\partial B}{\partial y^i}\frac{\partial B}{\partial y^j}\frac{\partial B}{\partial y^k}\frac{\partial B}{\partial y^l}\frac{\partial B}{\partial y^m}\right\}$$

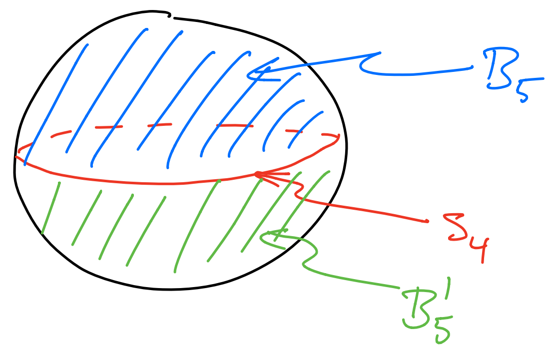
Gauss's theorem then gives:

$$I_{WZW}[U] = \frac{8\sqrt{2}\,n}{15\pi^2 F_\pi^5}\varepsilon^{\mu\nu\rho\sigma}\int_{S_4} d^4x\,\mathrm{Tr}\left\{B\frac{\partial B}{\partial x^\mu}\frac{\partial B}{\partial x^\nu}\frac{\partial B}{\partial x^\rho}\frac{\partial B}{\partial x^\sigma}\right\} + \mathcal{O}\left(\frac{B^6}{F_\pi^6}\right)$$

$\longrightarrow$ cannot be written as the integral of a chiral-invariant density over spacetime (every Goldstone boson would need to be accompanied by derivatives)!

Now to the coefficient n:

think of B_5 as half of a five-sphere S_5:



Could have used B_5':

$$I'_{WZW}[U] = -n \int_{B_5'} d^5y\, \omega(y)$$

boundary has opposite orientation

Path integral should be unaffected by choice of I_{WZW} and I'_{WZW}

$\rightarrow$ require

$$I_{WZW}[U] - I'_{WZW}[U] = n \underbrace{\int_{S_5} d^5y\, \omega(y)}_{=2\pi} = 2\pi \times \text{integer}$$

$\rightarrow$ n must be integer!